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ČERENKOV RADIATION FROM A CHARGED PARTICLE IN UNIFORM STRAIGHT MOTION THROUGH A PARTICULAR STRATIFIED MEDIUM

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Summary

A particular stratified medium is considered, constituted by a number of thin, plane, parallel and equispaced metallic films. The spacing between the films is filled with a perfect dielectric. A charged particle, in uniform straight motion goes through the structure, normally to the films. The velocity of the particle is above the Čerenkov threshold for the dielectric. The expression of the radiation is worked out as a function of different parameters. The radiation pattern is plotted in a number of particular cases.

ČERENKOV RADIATION FROM A CHARGED PARTICLE IN UNIFORM STRAIGHT MOTION THROUGH A PARTICULAR STRATIFIED MEDIUM.

1 - Introduction.

In a preceding Report ([1]) a particular stratified medium was considered, constituted by a certain number N of conducting and infinitely thin films, parallel to one another and equispaced by d, imbedded in a homogeneus medium with refractive index n (Fig.1.1).

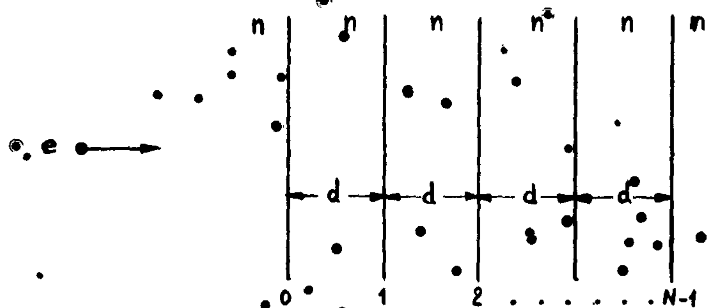

Fig.1.1- Cross section of the stratified medium considered in the below-threshold case.

A charged particle, in uniform straight motion, was assumed to impinge onto this structure, in a direction normal to the films.

The electromagnetic field radiated by the particle was expanded in a continuous set of time-harmonics. The gain over the case of a single film with infinite conductivity was evaluated as

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([1]) R.PRATESI, L.RONCHI, A.M.SCHEGGI, G.TORALDO di FRANCIA: Radiation from a Charged Particle in Uniform Straight Motion through a Particular Stratifield Medium - T.N. No.1, Grant 62-103 (Centro Microonde), 1962 - see also, Nuovo Cimento, 25, 756 (1962).

a function 1) of the wavenumber k, 2) of the parameters of the structure N, d, n, γ, where γ denotes surface conductivity of the films and 3) of the velocity of the particle $v = c\beta$. However the discussion was limited to the case below the Čerenkov threshold for the refractive index n, namely $\beta n < 1$.

In the present report we will consider the case above threshold

$$\beta n > 1 \tag{1.1}$$

The layered structure and the dielectric will be assumed to be bounded by empty space, both on the input and output side (Fig.1.2).

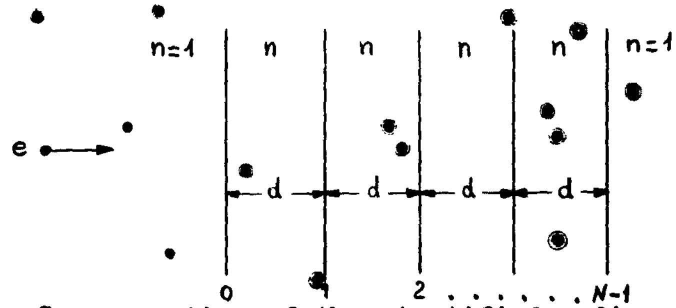


Fig.1.2 - Cross section of the stratified medium considered in the above-threshold case.

Such a system may be useful for the production of millimetric and submillimetric radiation, with better results than with a single film. The latter case has been recently reported in the Literature ([2]).

2 - The field of the particle in an infinite dielectric.

As is well known (see for example Ref.1), a charged

([2]) B.W.HAKKI, H.J.KRUMME: PROC.I.R.E., 49, 1334 (1961)

particle in uniform straight motion gives rise to a current which in cylindrical coordinates (ρ, z), is represented by

$$(2.1) \qquad \mathcal{J} = \frac{e}{\pi} \frac{\delta(\rho)}{\rho} \delta(\frac{z}{v} - t)$$

where e is the charge, z the coordinate along the path (z=0 for t=0). By expressing $\delta(z/v - t)$ as a Fourier integral, one finds for the harmonic corresponding to the frequency $\omega = ck$, the following expression:

$$(2.2) \qquad \mathcal{J}_k = \frac{ec}{\pi^2} \frac{\delta(\rho)}{\rho} \exp(ikz/\beta)$$

The usual convention is made of assuming the time-dependence $\exp(-i\omega t)$ and taking the real part of complex quantities.

The vector potential $A(\rho,z)$ due to harmonic (2.2) can be expressed as

$$(2.3) \qquad A(\rho,z) = \frac{eZ_o}{2\pi^2} f(\rho) \exp(ikz/\beta)$$

where Z_o is the free-space impedance and $f(\rho)$ satisfies the following equation

$$(2.4) \qquad \frac{d^2f}{d\rho^2} + \frac{1}{\rho}\frac{df}{d\rho} - (\frac{1}{\beta^2} - n^2)\, k^2 f = -2\,\frac{\delta(\rho)}{\rho}$$

This equation implies that f should present a logarithmic singularity for $\rho = 0$ [3].

---

[3] J.V. JELLEY ''Čerenkov Radiation and Its Applications'', Pergamon Press, New York 1958, p.17.

To begin with, in order to avoid divergent integrals, it will be expedient to consider the refractive index n between the films as a complex quantity, which will be denoted by $\tilde{n}$:

$$(2.5) \qquad \tilde{n} = n_r + i\, n_i$$

with n_r, n_i real and positive. This amounts to the same as assuming that the medium should present conductivity. The case with $\tilde{n}$ real will be obtained by making the conductivity or n_i to vanish. Accordingly, we will consider n_i as small with respect to unity.

In general, if $\tilde{n}$ is complex, the solution of (2.4) satisfying all requirements (radiation condition at infinity, logarithmic singularity at $\rho = 0$) may be equivalently expressed in terms of the modified Bessel function K_o or of the Hankel function of the first kind $H_o^{(1)}$. It will be expedient for us to use the K_o function, and write the solution of (2.4) in the form:

$$(2.6) \qquad f(\rho) = K_o(k\rho n\, \tau)$$

where τ is defined by

$$(2.7) \qquad \tau = \sqrt{\frac{1}{\beta^2 n^2} - 1} \qquad \mathrm{Re}(\tau) \geqslant 0$$

From (2.5) it follows that

$$(2.8) \qquad \mathrm{Im}(\tau) \leqslant 0$$

It will be noted that for $Im(n) \neq 0$, $Re(n\tau) > 0$. For $Im(n) = 0$, and $\beta\, Re(n) < 1$, τn equals the quantity σ introduced in Ref.1.

From (2.3), (2.6), by standard methods, one derives the expressions for the electromagnetic fields:

$$\underline{E} = \frac{eck\tau Z_o}{2\pi^2 \beta n} \left[K_1(kn\tau\rho)\underline{i}_\rho - in\tau\beta\, K_o(kn\tau\rho)\underline{i}_z \right] \exp(ikz/\beta)$$

(2.9)

$$\underline{H} = \frac{eck\tau n}{2\pi^2} K_1(kn\tau\rho)\, \exp(ikz/\beta)\underline{i}_\varphi$$

where K_1 denotes the modified Bessel function of first order, and $\underline{i}_\rho$, $\underline{i}_\varphi$, $\underline{i}_z$ represent unit vectors in the directions of the coordinate lines.

In the absence of the periodic structure described in sec.1, the total field is expressed by (2.9) with $n = 1$. For $n = 1$, the quantity τ turns out to be real and positive and will be denoted by τ_1. Thus the field in empty space would be expressed by

$$\underline{E}'(\rho,z) = \frac{eck\tau_1 Z_o}{2\pi^2\beta} \exp(ikz/\beta) \Big[K_1(k\tau_1\rho)\underline{i}_\rho -$$

(2.10)

$$- i\tau_1\beta\, K_o(k\tau_1\rho)\underline{i}_z \Big]$$

$$\underline{H}'(\rho,z) = \frac{eck\tau_1}{2\pi^2} \exp(ikz/\beta)\, K_1(k\tau_1\rho)\underline{i}_\varphi$$

The quantity τ for a dielectric with complex refractive index $\tilde{n}$, turns out to be complex, with positive real part and negative imaginary part. It will be denoted by $\tilde{\tau}$.

3 - The scattered field.

The scattered field, superimposed on the particle field, will be evaluated by taking into account the boundary conditions on each film.

The boundary conditions can be satisfied by assuming the scattered field to consist of two sets of progressive and regressive 'conical' TM-waves. In the general case of a complex refractive index n, a progressive conical TM-wave will be expressed by

$$(3.1)\qquad \begin{aligned} \underline{E}^p(\rho,z) &= P(\Theta)\exp(iknz\cos\Theta)\left[\cos\Theta\, J_1(kn\rho\sin\Theta)\,\underline{i}_\rho + \right. \\ &\left. \qquad + i\sin\Theta\, J_o(kn\rho\sin\Theta)\,\underline{i}_z\right] \\ Z_o\underline{H}^p(\rho,z) &= P(\Theta)\, n\exp(iknz\cos\Theta)\, J_1(kn\rho\sin\Theta)\,\underline{i}_\varphi \end{aligned}$$

and a regressive conical TM-wave will be expressed by

$$(3.2)\qquad \begin{aligned} \underline{E}^r(\rho,z) &= R(\Theta)\exp(-iknz\cos\Theta)\left[\cos\Theta\, J_1(kn\rho\sin\Theta)\,\underline{i}_\rho - \right. \\ &\left. \qquad - i\sin\Theta\, J_o(kn\rho\sin\Theta)\,\underline{i}_z\right] \\ Z_o\underline{H}^r(\rho,z) &= -R(\Theta)\, n\exp(-iknz\cos\Theta)\, J_1(kn\rho\sin\Theta)\,\underline{i}_\varphi \end{aligned}$$

where J_o, J_1 denote the Bessel functions of zero and first order respectively.

In both (3.1) and (3.2), Θ is either a real angle comprised between 0 and $\pi/2$, or a complex angle (evanescent waves) satisfying the following relations:

$$(3.3)\qquad \mathrm{Re}(n\cos\Theta) \geqslant 0, \qquad \mathrm{Im}\,(n\cos\Theta) \geqslant 0$$

In order to satisfy the boundary conditions, we will as-

sume that:

1) before the input surface ($z < 0$) the total field is given by the particle field $\underline{E}'$, $\underline{H}'$ (2.10), plus a set of regressive conical waves. Precisely:

$$
(3.4) \qquad \begin{aligned} \underline{E}(\rho,z) &= \underline{E}'(\rho,z) + \underline{E}^r_o(\rho,z) \\ \underline{H}(\rho,z) &= \underline{H}'(\rho,z) + \underline{H}^r_o(\rho,z) \end{aligned}
$$

where, according to (3.2),

$$
(3.5) \qquad \begin{aligned} \underline{E}^r_o &= \underline{i}_\rho \int_{C_o} R_o(\Theta') \cos\Theta' \exp(-ikz\cos\Theta')\, J_1(k\rho\sin\Theta')\, d\Theta' - \\ &- i\, \underline{i}_z \int_{C_o} R_o(\Theta') \sin\Theta' \exp(-ikz\cos\Theta')\, J_o(k\rho\sin\Theta')\, d\Theta', \\ Z_o \underline{H}^r_o &= -\underline{i}_\varphi \int_{C_o} R_o(\Theta') \exp(-ikz\cos\Theta')\, J_1(k\rho\sin\Theta')\, d\Theta' \end{aligned}
$$

with the conditions

$$
\mathrm{Re}(\cos\Theta') \geqslant 0 \quad , \quad \mathrm{Im}(\cos\Theta') \geqslant 0 .
$$

The path of the integration C_o will be chosen in a suitable way.

2) after the output surface ($z > (N-1)\,d$), the total field is given by the particle field $\underline{E}'$, $\underline{H}'$, expressed by (2.10), plus a set of progressive conical waves. Precisely:

$$
(3.6) \qquad \begin{aligned} \underline{E}(\rho,z) &= \underline{E}'(\rho,z) + \underline{E}^p_N(\rho,z) \\ \underline{H}(\rho,z) &= \underline{H}'(\rho,z) + \underline{H}^p_N(\rho,z) \end{aligned}
$$

where, according to (3.1),

$$\underline{E}^p_N = \underline{i}_\rho \int_{C_{N-1}} P_N(\Theta') \cos\Theta' \exp(ikz\cos\Theta')\, J_1(k\rho\sin\Theta')\, d\Theta' +$$

$$(3.7) \qquad + i\,\underline{i}_z \int_{C_{N-1}} P_N(\Theta') \sin\Theta' \exp(ikz\cos\Theta')\, J_0(k\rho\sin\Theta')\, d\Theta' ,$$

$$Z_o \underline{H}^p_N = \int_{C_{N-1}} P_N(\Theta') \exp(ikz\cos\Theta')\, J_1(k\rho\sin\Theta')\, d\Theta'\, \underline{i}_\varphi$$

with the conditions

$$\mathrm{Re}(\cos\Theta') \geqslant 0 \qquad , \qquad \mathrm{Im}(\cos\Theta') \geqslant 0$$

and suitable path C_{N-1}

3) the field between any two consecutive films, say the films number $m-1$ and number m, is constituted by the particle field $\underline{E}''$, $\underline{H}''$ given by (2.9) with $n = \tilde{n}$, $\tau = \tilde{\tau}$, plus a set of progressive and a set of regressive conical waves. Precisely, for $(m-1)\,d < z < md$,

$$(3.8) \qquad \begin{aligned} \underline{E}(\rho,z) &= \underline{E}''(\rho,z) + \underline{E}_{m-1,m}(\rho,z) \\ \underline{H}(\rho,z) &= \underline{H}''(\rho,z) + \underline{H}_{m-1,m}(\rho,z) \end{aligned}$$

with

$$(3.9) \qquad \begin{aligned} \underline{E}_{m-1,m} &= \underline{E}^p_m + \underline{E}^r_m \\ \underline{H}_{m-1,m} &= \underline{H}^p_m + \underline{H}^r_m \end{aligned}$$

and

$$\underline{E}^{p}_{m} = \underline{i}_{\rho} \int_{C_m} P_m(\theta) \cos\theta \exp(ik\tilde{n}z \cos\theta)\, J_1(k\tilde{n}\rho \sin\theta)\, d\theta \; +$$

$$(3.10) \qquad + \, i \, \underline{i}_{z} \int_{C_m} P_m(\theta) \sin\theta \exp(ik\tilde{n}z \cos\theta)\, J_0(k\tilde{n}\rho \sin\theta)\, d\theta \, ,$$

$$Z_o \underline{H}^{p}_{m} = \tilde{n}\, \underline{i}_{\varphi} \int_{C_m} P_m(\theta) \exp(ik\tilde{n}z \cos\theta)\, J_1 (k\tilde{n}\rho \sin\theta)\, d\theta$$

$$\underline{E}^{r}_{m} = \underline{i}_{\rho} \int_{C'_m} R_m(\theta) \cos\theta \exp(-ik\tilde{n}z \cos\theta)\, J_1(k\tilde{n}\rho \sin\theta)\, d\theta \; -$$

$$(3.11) \qquad - \, i \, \underline{i}_{z} \int_{C'_m} R_m(\theta) \sin\theta \exp(-ik\tilde{n}z \cos\theta)\, J_0(k\tilde{n}\rho \sin\theta)\, d\theta \, ,$$

$$Z_o \underline{H}^{r}_{m} = -\tilde{n}\, \underline{i}_{\varphi} \int_{C'_m} R_m(\theta) \exp(-ik\tilde{n}z \cos\theta)\, J_1(k\tilde{n}\rho \sin\theta)\, d\theta$$

Conditions (3.3) are now expressed by

$$\mathrm{Re}(\tilde{n} \cos\theta) \geqslant 0 \quad , \qquad \mathrm{Im}(\tilde{n} \cos\theta) \geqslant 0$$

Thus the problem of satisfying the boundary conditions reduces to determining the complex amplitudes R_o, P_N, R_m, P_m, as well as the integration paths C_o, C_{N-1}, C_m, C'_m.

By denoting with subscript t the tangential components, and by taking into account (3.4), (3.6), (3.8), the boundary conditions read at $z = 0$:

$$(3.12) \qquad \begin{aligned} (\underline{E}')_t + (\underline{E}^{r}_{o})_t &= (\underline{E}'')_t + (\underline{E}_{o,1})_t \\ (\underline{H}')_t + (\underline{H}^{r}_{o})_t - (\underline{H}'')_t - (\underline{H}_{o,1})_t &= \gamma \left[(\underline{E}')_t + (\underline{E}^{r}_{o})_t \right] \end{aligned}$$

at $z = md$, $(m \neq 0,\ m \neq N-1)$:

$$(3.13)\qquad \begin{aligned} &(\underline{E}'')_t + (\underline{E}_{m-1,m})_t = (\underline{E}'')_t + (\underline{E}_{m,m+1})_t \\ &(\underline{H}'')_t + (\underline{H}_{m-1,m})_t - (\underline{H}'')_t - (\underline{H}_{m,m+1})_t = \gamma\left[(\underline{E}'')_t + (\underline{E}_{m-1,m})_t\right] \end{aligned}$$

at $z = (N-1)d$:

$$(3.14)\qquad \begin{aligned} &(\underline{E}'')_t + (\underline{E}_{N-2,N-1})_t = (\underline{E}')_t + (\underline{E}^p_N)_t \\ &(\underline{H}'')_t + (\underline{H}_{N-2,N-1})_t - (\underline{H}')_t - (\underline{H}^p_N)_t = \gamma\left[(\underline{E}')_t + (\underline{E}^p_N)_t\right] \end{aligned}$$

Let us first consider Eqs.(3.13) which by (3.9) can be rewritten as

$$(3.15)\qquad \begin{aligned} &(\underline{E}^p_m)_t + (\underline{E}^r_m)_t = (\underline{E}^p_{m+1})_t + (\underline{E}^r_{m+1})_t \\ &(\underline{H}^p_m)_t + (\underline{H}^r_m)_t - (\underline{H}^p_{m+1})_t - (\underline{H}^r_{m+1})_t = \gamma\left[(\underline{E}'')_t + (\underline{E}^p_m)_t + (\underline{E}^r_m)_t\right] \end{aligned}$$

In order to suitably transform (3.15) we will use the formula [4]

$$(3.16)\qquad \int_0^\infty \frac{x^{\nu+1}\, J_\nu(ax)}{(x^2+z^2)^{m+1}}\, dx = \frac{a^m\, z^{\nu-m}\, K_{\nu-m}(az)}{2^m\, m!}$$

which holds for $a > 0$, $\mathrm{Re}(z) > 0$ and $-1 < \mathrm{Re}(\nu) < 2m + 3/2$.

---

[4] G.N.WATSON: Theory of Bessel Functions, (Cambridge University Press, 1958), p. 425, Eq.4.

By putting $\nu = 1$, $m = 0$, $a = k\rho$, $z = \tilde{n}\tilde{\tau}$, $x = \sin\theta'$, Eq.(3.16) gives

$$(3.17)\quad K_1(k\tilde{n}\tilde{\tau}\rho) = \frac{1}{\tilde{n}\tilde{\tau}} \int_0^\infty \frac{\sin^2\theta' \, J_1(k\rho\sin\theta')}{\sin^2\theta' + \tilde{n}^2\tilde{\tau}^2} \, d\sin\theta'$$

If in (3.17) we change the variable $\sin\theta'$ into the variable θ defined by

$$(3.18)\quad \sin\theta = \frac{1}{\tilde{n}}\sin\theta' \quad , \quad \mathrm{Re}(\tilde{n}\cos\theta) \geqslant 0 \quad , \quad \mathrm{Im}(\tilde{n}\cos\theta) \geqslant 0$$

the new integration path in the complex θ-plane becomes the line C defined by (3.18) and $\sin\theta' > 0$. By putting $\theta = x + iy$, it is an easy matter to find that the equation of path C in the xy plane is

$$(3.19)\qquad \tan x \coth y = - \frac{n_r}{n_i}$$

The behavior of C is shown in Fig.3.1., where $x_0 = \arctan(n_r/n_i)$ represents an asymptote, and x_0 is the angle made by C with the y - axis at O.

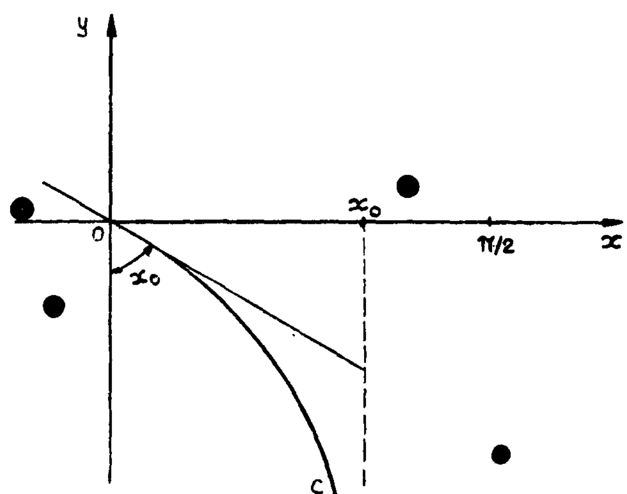


Fig.3.1 - The integration path C

Thus Eq.(3.17) may be rewritten as

$$K_1(k\tilde{n}\,\tilde{\tau}\,\rho) = \frac{1}{\tilde{\tau}} \int_C \frac{\sin^2\theta}{\sin^2\theta + \tilde{\tau}^2}\, J_1(k\rho\tilde{n}\sin\theta)\cos\theta\, d\theta \qquad (3.20)$$

Now both paths C_m and C'_m appearing in (3.10) (3.11) will be taken to coincide with C for all values of $m \neq 0, N-1$. Then Eqs. (3.15) turn out to be satisfied when the following relations are satisfied:

$$\begin{aligned} & P_m \zeta^{m} + R_m \zeta^{-m} = P_{m+1}\zeta^{m} + R_{m+1}\zeta^{-m} \\ & P_m \zeta^{m} - R_m \zeta^{-m} - P_{m+1}\zeta^{m} + R_{m+1}\zeta^{-m} = 2\delta\left[P_m\zeta^{m} + R_m\zeta^{-m} + F\varepsilon^{m}\right] \end{aligned} \qquad (3.21)$$

where we put

$$\zeta = \exp(i\psi) \quad , \quad \psi = kd\tilde{n}\cos\theta \qquad (3.22)$$

$$\delta = \frac{\gamma Z_o}{2\tilde{n}}\cos\theta \qquad (3.23)$$

$$\tilde{F} = \frac{eck\ Z_o}{2\pi^2\beta\tilde{n}}\ \frac{\sin^2\theta}{\sin^2\theta + \tilde{\tau}^2} \qquad (3.24)$$

$$\varepsilon = \exp(i\phi) \quad , \quad \phi = kd/\beta \qquad (3.25)$$

Relations (3.21) are formally equal to the analogous relations found in Ref.1. By introducing the quantities (see Ref.1)

$$\begin{aligned} p_m &= \delta\zeta^{m-1} P_m + \left[\alpha_1 - \zeta(1-\delta)\right]\zeta^{-m} R_m \\ r_m &= \delta\zeta^{m-1} P_m + \left[\alpha_2 - \zeta(1-\delta)\right]\zeta^{-m} R_m \end{aligned} \qquad (3.26)$$

with

$$(3.27)\quad \alpha_1 = \frac{1}{\alpha_2} = \cos\psi - i\,\delta\,\sin\psi + i\sqrt{(1+\delta^2)\,\sin^2\psi + i\,\delta\,\sin 2\psi}$$

Eqs.(3.21) transform into:

$$(3.28)\quad \begin{aligned} p_{m+1} &= \alpha_1\, p_m - \delta\,\tilde{F}\,(1-\alpha_1\,\zeta^{-1})\,\varepsilon^m \\ r_{m+1} &= \alpha_2\, r_m - \delta\,\tilde{F}\,(1-\alpha_2\,\zeta^{-1})\,\varepsilon^m \end{aligned}$$

Then, recalling that (3.28) hold for m varying from 1 to N-2 one obtains,

$$(3.29)\quad \begin{aligned} p_{N-1} &= \alpha_1^{N-2}\,p_1 - \delta\,\varepsilon\,\tilde{F}\,(1-\alpha_1\,\zeta^{-1})\,\frac{\varepsilon^{N-2}-\alpha_1^{N-2}}{\varepsilon-\alpha_1} \\ r_{N-1} &= \alpha_2^{N-2}\,r_1 - \delta\,\varepsilon\,\tilde{F}\,(1-\alpha_2\,\zeta^{-1})\,\frac{\varepsilon^{N-2}-\alpha_2^{N-2}}{\varepsilon-\alpha_2} \end{aligned}$$

Let us now consider Eqs. (3.12).

Quantities $(\underline{E}')_t$, $(\underline{H}')_t$ are given by (2.10); $(\underline{E}'')_t$, $(\underline{H}'')_t$ by (2.9); $(\underline{E}_o^r)_t$, $(\underline{H}_o^r)_t$ by (3.5), and finally, $(\underline{E}_{o,1})_t$, $(\underline{H}_{o,1})_t$ by (3.10), (3.11), through (3.9), for m = 0.

The function K_1 appearing in (2.9) will be transformed by using (3.20). In order to write an analogous expression for the function K_1 appearing in (2.10), we will put in (3.16) $\nu = 1$, $m = 0$, $a = k\rho$, $z = \tau_1$, $x = \sin\Theta'$, and obtain:

$$(3.30)\quad K_1(k\tau_1\rho) = \frac{1}{\tau_1}\int_0^\infty \frac{\sin^2\Theta'}{\sin^2\Theta' + \tau_1^2}\; J_1(k\rho\,\sin\Theta')\; d\sin\Theta'$$

With the transformation

$$\sin\theta = \frac{1}{\tilde{n}}\sin\theta' \quad , \quad \mathrm{Re}(\tilde{n}\cos\theta) \geqslant 0 \quad , \quad \mathrm{Im}(\tilde{n}\cos\theta) \geqslant 0$$

the right-hand side of (3.30) becomes

$$(3.31) \quad K_1(k\,\tau_1\rho) = \frac{1}{\tau}\int_C \frac{\sin^2\theta}{\sin^2\theta + \tau^2}\, J_1(k\tilde{n}\rho\sin\theta)\cos\theta\, d\theta$$

where C is the same path defined by (3.18), and

$$(3.32) \qquad \tau = \frac{1}{\tilde{n}}\,\tau_1$$

Analogously, in Eqs.(3.5) we will change the variable θ' into the variable θ defined by

$$(3.33) \qquad \sin\theta' = \tilde{n}\sin\theta$$

and choose C_o in such a way that the corresponding integration path in the θ-plane coincide with C. C_o turns out to be the path represented in Fig.3.2.

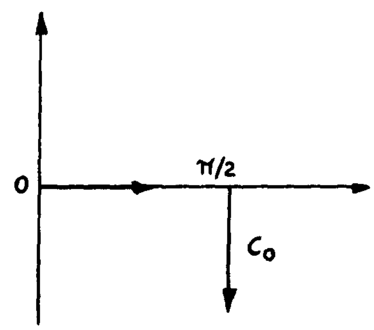


Fig.3.2 - The integration path C_o

Thus Eqs.(3.12) turn out to be satisfied if the following relations are satisfied:

$$\text{(3.34)}\quad \begin{aligned} & F + \tilde{n} R_0 = \tilde{F} + P_1 + R_1 \\ & \beta\tilde{n} \cos\theta \left[(F/\tilde{n}^2) - \tilde{F} \right] - \eta R_0 - P_1 + R_1 = 2\,\delta\,(\tilde{n} R_0 + F) \end{aligned}$$

where

$$\text{(3.35)}\qquad F = \frac{eck\, Z_0 \tilde{n}}{2\pi^2 \beta} \quad \frac{\sin^2\theta}{\sin^2\theta + \tau^2}$$

$$\text{(3.36)}\qquad \eta = \frac{\cos\theta}{\cos\theta'}$$

and $\cos\theta' = \sqrt{1 - \tilde{n}^2 \sin^2\theta}$, $\mathrm{Re}(\cos\theta') \geqslant 0$, $\mathrm{Im}(\cos\theta') \geqslant 0$.
From (3.34) one derives the expressions of P_1, R_1 in terms of R_0. Then the quantities p_1, r_1 defined by (3.26) for m=1, turn out to be given by

$$\text{(3.37)}\quad \begin{aligned} p_1 &= \left[\delta A_0 + \left\{ \alpha_1 \zeta^{-1} - (1-\delta) \right\} C_0 \right] + \left[\delta B_0 + \left\{ \alpha_1 \zeta^{-1} - (1-\delta) \right\} D_0 \right] R_0 \\ r_1 &= \left[\delta A_0 + \left\{ \alpha_2 \zeta^{-1} - (1-\delta) \right\} C_0 \right] + \left[\delta B_0 + \left\{ \alpha_2 \zeta^{-1} - (1-\delta) \right\} D_0 \right] R_0 \end{aligned}$$

where

$$\text{(3.38)}\quad \begin{aligned} A_0 &= \frac{1}{2} \left\{ \beta\tilde{n} \cos\theta \left(\frac{F}{\tilde{n}^2} - \tilde{F} \right) + (1-2\delta) F - \tilde{F} \right\} \\ B_0 &= \frac{1}{2} \left\{ (1-2\delta)\, \tilde{n} - \eta \right\} \end{aligned}$$

$$C_o = F - \tilde{F} - A_o$$

(3.38)

$$D_o = \tilde{n} - B_o$$

Analogously, from (3.14) one derives

$$\varepsilon^{N-1}\tilde{F} + P_{N-1}\zeta^{N-1} + R_{N-1}\zeta^{-(N-1)} = \varepsilon^{N-1}F + \tilde{n}P_N\zeta_1^{N-1},$$

(3.39) $$-\beta\tilde{n}\cos\theta\,\varepsilon^{N-1}\left(\frac{F}{\tilde{n}^2} - \tilde{F}\right) + P_{N-1}\zeta^{N-1} - R_{N-1}\zeta^{-(N-1)} -$$

$$-\eta P_N\zeta_1^{N-1} = 2\delta\left[F\varepsilon^{N-1} + \tilde{n}P_N\zeta_1^{N-1}\right]$$

with

(3.40) $$\zeta_1 = \exp(i\,\psi_1) \quad , \quad \psi_1 = k\,d\cos\theta'$$

From (3.39), (3.26), one finds

(3.41)
$$p_{N-1} = \left\{\delta\zeta^{-1}A_1 + \left[\alpha_1 - \zeta(1-\delta)\right]C_1\right\} + \left\{\delta\zeta^{-1}B_1 + \left[\alpha_1 - \zeta(1-\delta)\right]D_1\right\}P_N\zeta_1^{N-1}$$
$$r_{N-1} = \left\{\delta\zeta^{-1}A_1 + \left[\alpha_2 - \zeta(1-\delta)\right]C_1\right\} + \left\{\delta\zeta^{-1}B_1 + \left[\alpha_2 - \zeta(1-\delta)\right]D_1\right\}P_N\zeta_1^{N-1}$$

with

(3.42)
$$A_1 = \varepsilon^{N-1}(A_o + 2\delta F)$$
$$B_1 = D_o$$
$$C_1 = \varepsilon^{N-1}(C_o - 2\delta F)$$
$$D_1 = B_o$$

Finally, combining (3.29),(3.37),(3.41), with some laborious manipulations, one finds:

$$
\begin{aligned}
R_0 = {} & \frac{1}{Q(\alpha_1^N-\alpha_2^N)-A(\alpha_1^{N-1}-\alpha_2^{N-1})}\cdot\frac{1}{2} \\
& \times\Big\{ 2\delta\tilde{F}(M-\alpha_1\tilde{n})\,\frac{\varepsilon^N-\alpha_1^N}{\varepsilon-\alpha_1} - 2\delta\tilde{F}(M-\alpha_2\tilde{n})\,\frac{\varepsilon^N-\alpha_2^N}{\varepsilon-\alpha_2} + \\
& +\beta\tilde{n}\cos\Theta\Big(\tilde{F}-\frac{F}{\tilde{n}^2}\Big)\Big[M(\alpha_1^{N-1}-\alpha_2^{N-1})+\tilde{n}\,\varepsilon^{N-1}(\alpha_1-\alpha_2)-\tilde{n}(\alpha_1^N-\alpha_2^N)\Big]+ \\
& +(F-\tilde{F})\Big[M_1(\alpha_1^{N-1}-\alpha_2^{N-1})+\eta\,\varepsilon^{N-1}(\alpha_1-\alpha_2)-(\eta+2\delta\tilde{n})\,(\alpha_1^N-\alpha_2^N)\Big]\Big\}
\end{aligned}
\tag{3.43}
$$

where

$$
\begin{aligned}
A &= \zeta W^2-\zeta^{-1}W_1^2 \\
Q &= (W+W_1)\Big[W-W_1+\delta(W+W_1)\Big] \\
W &= \frac{1}{2}(\tilde{n}+\eta)\quad ; \quad W_1=\frac{1}{2}(\tilde{n}-\eta) \\
M &= \zeta W+\zeta^{-1}W_1 \\
M_1 &= \zeta W-\zeta^{-1}W_1
\end{aligned}
\tag{3.44}
$$

An analogous expression is found for P_N. It is to be noted that, considering R_0 and P_N as functions of β, the following relation holds:

$$
P_N(\beta) = -(\zeta_1^{-1}\varepsilon)^{N-1}R_0(-\beta)
\tag{3.45}
$$

It was to be expected that $P_N(\beta)$ and $R_0(-\beta)$ should have equal modulus since opposite values of β correspond to the particle traveling into opposite directions (see also Ref.1).

As a consequence we can limit ourselves to discussing the values of R_0 for both positive and negative values of β.

To summarise, the back-scattered field is given by (3.5) where $R_0(\Theta')$ is given by (3.43), with Θ expressed in terms of Θ' according to (3.33), and C_0 is the path shown in Fig.3.2.

4.1- Asymptotic expression of the scattered field.

Let us first consider the expression (3.5) of the magnetic field

$$(4.1\text{-}1)\quad H_0^r = -\frac{1}{Z_0}\int_{C_0} R_0(\Theta')\exp(-ikz\cos\Theta')\,J_1(k\rho\sin\Theta')\,d\Theta'$$

The case $\rho = 0$ must be treated as a limit, because the treatment cannot be applied for $\rho = 0$ due to the condition $a > 0$ for aplyability of (3.16). Let us now put (Fig.4.1)

$$(4.1\text{-}2)\quad \begin{aligned} \rho &= r\sin\Theta \\ z &= -r\cos\Theta \end{aligned}$$

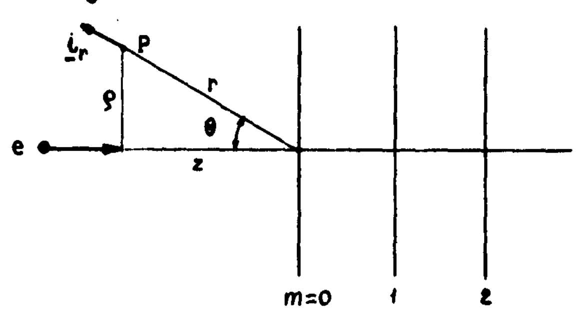


Fig.4.1 - The reference system

and proceed to the evaluation of (4.1-1) for a fixed value of $\Theta > 0$ and $r \to \infty$.

Since $\sin\Theta'$ is real and positive all along the integration path, one can substitute (*) for J_1 into (4.1-1) the first term of its asymptotic expression obtaining

$$(4.1\text{-}3) \quad H_0^r = H_0^r(r,\Theta) = e^{-i\pi/4}\, I_1(r,\Theta) - e^{i\pi/4}\, I_2(r,\Theta)$$

where

$$(4.1\text{-}4) \quad I_1 = \frac{1}{Z_0} \frac{i}{\sqrt{2\pi k\, r \sin\Theta}} \int_{C_0} \frac{R_0(\Theta')}{\sqrt{\sin\Theta'}} \exp\left[ikr\cos(\Theta - \Theta')\right] d\Theta'$$

$$(4.1\text{-}5) \quad I_2 = \frac{1}{Z_0} \frac{i}{\sqrt{2\pi kr \sin\Theta}} \int_{C_0} \frac{R_0(\Theta')}{\sqrt{\sin\Theta'}} \exp\left[ikr\cos(\Theta + \Theta')\right] d\Theta'$$

with $\quad \sqrt{\sin\Theta} > 0 \quad , \quad \sqrt{\sin\Theta'} \geqslant 0$

Of course the far E_0^r field can be immediately obtained from H_0^r, as can also be seen from (3.5) (wave zone).

As long as we are only interested in the far field, the integration path C_0 in (4.1-4)(4.1-5) can be replaced by its real

---

(*) To be precise, the asymptotic expression of $J_1(kr\sin\Theta.\sin\Theta')$ is not appliable for $\sin\Theta' = 0$. However, it may be shown that the error introduced by using it all along the integration path may be made arbitrarily small. This is essentially due to the fact that the quantity $R_0(\Theta')$ has a second-order zero at $\Theta' = 0$, as will be seen later. Accordingly, if one considers a small portion of C_0 from $\Theta' = 0$ to $\Theta' = \varepsilon$, its contribution to the whole integral is negligible, even if the asymptotic expression is substituted for J_1.

portion from 0 to $\pi/2$, since the portion from $\pi/2$ to $\pi/2-i\infty$ corresponds to an evanescent field.

Then the integrals (4.1-4), (4.1-5) may be evaluated by applying the principle of stationary phase. However, one has first to investigate if there are singularities of the integrands on the path of the integration.

The singularities of interest are the singularities at finite distance of the function

$$(4.1\text{-}6) \qquad \frac{R_0(\theta')}{\sqrt{\sin\theta'}} \quad , \quad \sqrt{\sin\theta'} \geqslant 0$$

It will be useful to remember that the quantity $\sin\theta'$ is real and positive along the integration path, and that the quantity $\cos\theta$, appearing in Eq.(3.43) both explicitly and implicitly through the functions δ, γ, F, $\tilde{F}$, α_1, α_2, is defined by

$$(4.1\text{-}7) \qquad \tilde{n}\sin\theta = \sin\theta' \quad , \quad \mathrm{Re}(\tilde{n}\cos\theta) \geqslant 0 \quad , \quad \mathrm{Im}(\tilde{n}\cos\theta) \geqslant 0$$

4.2 - Branch points.

According to (4.1-6) $\theta' = 0$ is a branch point. Now, from (3.43), one sees that R_0 may be written as a term with F as a factor plus another term with $\tilde{F}$ as a factor. For $\tau \neq 0$, $\tilde{\tau} \neq 0$, both F and $\tilde{F}$ contain the factor $\sin^2\theta$ (see (3.24) and (3.35)), which, according to (4.1-7), is proportional to $\sin^2\theta'$. Consequently, $\theta' = 0$ is a second-order zero for $R_0(\theta')$. It follows, as already noted (see footnote at p.19), that the singularity at $\theta' = 0$ due to the factor $(\sin\theta')^{-1/2}$ of function (4.1-6) is immaterial for our discussion.

As to the other branch points of the integrands, which correspond to $\cos\theta = 0$, that is to $\sin\theta' = \pm\tilde{n}$, they are located

outside the integration path and all we have to do is to choose the branch which corresponds to (4.1-7).

4.3 - <u>Poles</u>.

In order to investigate the poles of $R_o(\Theta')$, one has to investigate the poles of the functions F and $\tilde{F}$, defined by (3.35) and (3.24) respectively, and the zeros of the quantity

$$(4.3\text{-}1) \qquad Q(\alpha_2^N - \alpha_1^N) - A(\alpha_2^{N-1} - \alpha_1^{N-1})$$

The value $\Theta' = \pi/2$ is a first-order pole of the numerator of R_o, but, at the same time, it is a second-order pole of the denominator, so that, as a whole, $\Theta' = \pi/2$ is a zero of R_o.

Let us now study the poles of F. Recalling (3.35) and (4.1-7), we can write

$$(4.3\text{-}2) \qquad F(\Theta') = \frac{eckZ_o\tilde{n}}{2\pi^2\beta} \frac{\sin^2\Theta'}{\sin^2\Theta' + \tau_1^2}$$

Hence, the poles of $F(\Theta')$ are solutions of the equation

$$\sin^2\Theta' = -\tau_1^2 = 1 - \frac{1}{\beta^2}$$

and since $1/\beta^2 > 1$, no one of these poles is located on C_o.

The poles of $\tilde{F}$ are analogously found to be of the first order, given by the solution of the equation

$$(4.3\text{-}5) \qquad \cos^2\Theta = \frac{1}{\beta^2\tilde{n}^2} = \frac{1}{\beta^2|\tilde{n}|^4}(n_r^2 - n_i^2 - 2i\, n_r n_i)$$

For $n_i \neq 0$, the solution of (4.3-5) closest to C_o has positive real

and imaginary parts. For $n_i \to 0$, this solution tends to a point of C_o, precisely to the value of Θ' corresponding to

$$\Theta = \arccos(1/\beta n_r) \tag{4.3-6}$$

However, the function of Θ' multiplying $\tilde{F}$ in the expression of R_o has a first order zero at the same value of Θ' so that $R_o(\Theta')$ is regular at the pole of $\tilde{F}$. This fact allows us to remove the assumption $n_i \neq 0$ and to consider directly the limiting case $n_i = 0$.

It remains to examine the zeros of the function (4.3-1). Recalling that $\alpha_1\alpha_2 = 1$ (see Eq.(3.27)), one can put

$$\alpha_1 = e^{i\chi} \quad , \qquad \alpha_2 = e^{-i\chi} \tag{4.3-7}$$

with

$$\cos\chi = \cos\psi - i\,\delta\sin\psi \tag{4.3-8}$$

$$\sin\chi = \sqrt{(1+\delta^2)\sin^2\psi + 2i\,\delta\sin\psi\cos\psi} \tag{4.3-9}$$

Without loss of generality, one can choose

$$\mathrm{Re}(\sin\chi) \geqslant 0 \tag{4.3-10}$$

since, by choosing $\mathrm{Re}(\sin\chi) < 0$, one simply interchanges α_1 and α_2, with no variation for $R_o(\Theta')$. With position (4.3-7), the zeros of the function (4.3-1) are the solutions of the equation

$$Q\sin N\chi = A\sin(N-1)\chi \tag{4.3-11}$$

A simple root corresponds to $\sin\chi = 0$. However, $\sin\chi = 0$ yields $\alpha_1 = \alpha_2$, and for $\alpha_1 = \alpha_2$ the numerator of R_0 vanishes, too. Accordingly, there are no poles of R_0 at the values of Θ' corresponding to $\sin\chi = 0$.

It is not an easy matter to solve (4.3-11) for Θ'. However, we need only to know if there are solutions of (4.3-11) in the range $0 \leq \Theta' \leq \pi/2$ for $n_i = 0$. We will show that there are no such solutions.

Eq.(4.3-11) is easily transformed into

$$\tan(N-1)\chi = - \frac{Q \sin\chi}{Q \cos\chi - A} \tag{4.3-12}$$

It can now be shown that for $0 \leq \Theta' \leq \pi/2$ the imaginary parts of the left-hand and right-hand sides have always opposite signs.

Let us put $\chi = a + ib$. It is easily proved that $\mathrm{Im}\left[\tan(N-1)\chi\right]$ has the same sign as b.

As regards the right-hand side of (4.3-12), the sign of its imaginary part is the same as the sign of

$$\mathrm{Im}\left[-Q \sin\chi\,(Q^{*} \cos\chi^{*} - A^{*})\right]$$

From (3.44) one can derive that Q is real for n real, so that one can write

$$\begin{aligned} &\mathrm{Im}\left[-Q \sin\chi\,(Q^{*} \cos\chi^{*} - A^{*})\right] = \\ &= Q^2\,\mathrm{Re}(\sin\chi)\,\mathrm{Im}(\cos\chi) - Q^2\,\mathrm{Im}(\sin\chi)\,\mathrm{Re}(\cos\chi) - \\ &- Q\,\mathrm{Re}(\sin\chi)\,\mathrm{Im}(A) + Q\,\mathrm{Im}(\sin\chi)\,\mathrm{Re}(A) \end{aligned} \tag{4.3-13}$$

From (3.33), (3.23), (3.22) there follows that $\sin\Theta$ is real and smaller than 1, and therefore $\cos\Theta$ is real and positive; ψ is real and positive; δ is real. Accordingly, from (4.3-8) one has

$$\mathrm{Re}(\cos\chi) = \cos a \cosh b = \cos\psi \tag{4.3-14}$$

$$\mathrm{Im}(\cos\chi) = -\sin a \sinh b = -\delta \sin\psi \tag{4.3-15}$$

One the other hand, one can write

$$\mathrm{Re}(\sin\chi) = \sin a \cosh b \tag{4.3-16}$$

$$\mathrm{Im}(\sin\chi) = \cos a \sinh b \tag{4.3-17}$$

Finally, from (3.44) one has

$$\mathrm{Re}(A) = (W^2 - W_1^2)\cos\psi = n_r \eta \cos\psi \tag{4.3-18}$$

$$\mathrm{Im}(A) = (W^2 + W_1^2)\sin\psi = \tfrac{1}{2}(n_r^2 + \eta^2)\sin\psi \tag{4.3-19}$$

By introducing (4.3-14) to (4.3-19) into (4.3-13), one finds

$$\begin{aligned} &\mathrm{Im}\left[-Q\sin\chi(Q\cos\chi^* - A^*)\right] = \\ &= -Q\frac{\sinh b}{\cosh b}\left[Q\cosh^2 b - n_r\eta\cos^2\psi\right] \\ &\quad - \frac{1}{2}Q\delta\frac{\cosh b}{\sinh b}\left[n_r^2 + \eta^2\right]\sin^2\psi \end{aligned} \tag{4.3-20}$$

With the help of (3.44) one derives that

$$Q \cosh^2 b \geqslant n_r \eta \cos^2 \psi$$

Accordingly, and recalling that Q is positive, from (4.3-20) one derives that in general, the imaginary part of the right-hand side of (4.3-12) has a sign opposite to b, while, as already noted, the imaginary part of the left-hand side has the same sign as b. The particular case b = 0 needs not be considered, because it corresponds to $\sin \gamma = 0$ and therefore to $\alpha_1 = \alpha_2$.

As a conclusion, there are no poles of $R_o(\Theta')$ in the range $0 \leqslant \Theta' \leqslant \pi/2$, for $n_i = 0$.

4.4 - The asymptotic expressions of the fields.

Turning back to (4.1-4), and applying the principle of stationary phase, we get

$$(5.4\text{-}1) \qquad I_1 \sim i\, e^{-i\pi/4} \frac{R_o(\Theta)}{Z_o k \sin\Theta} \frac{e^{ikr}}{r}$$

As to I_2, defined by (4.1-5), its main term is of the order of $r^{-3/2}$, since the 'stationary' point is outside C_o. Accordingly, it may be neglected compared with I_1. Thus, by introducing (4.4-1) into (4.1-3), one obtains

$$(4.4\text{-}2) \qquad \underline{H}_o^r(r,\Theta) \sim \frac{1}{Z_o k} \frac{R_o(\Theta)}{\sin\Theta} \frac{e^{ikr}}{r} \; \underline{i}_\varphi$$

and then

$$(4.4\text{-}3) \qquad \underline{E}_o^r(r,\Theta) \; \frac{1}{k} \frac{R_o(\Theta)}{\sin\Theta} \frac{e^{ikr}}{r} \left(\underline{i}_\varphi \times \underline{i}_r \right)$$

Then, by standard methods (see e.g. Ref.1), one obtains the following expression for the energy dU radiated in the solid angle $d\Omega = 2\pi \sin\Theta \, d\Theta$ and in the frequency range k, k+dk:

$$(4.4\text{-}4) \qquad dU = \frac{\pi}{cZ_o k^2} \frac{|R_o(\Theta)|^2}{\sin^2\Theta} \, dk \, d\Omega$$

It is an easy matter to verify that for $\Theta \to 0$ one has $dU \to 0$.

5 - Some numerical results.

Eq.(3.43) introduced into (4.4-4) allows to evaluate the energy radiated per unit solid angle and per unit frequency range, for any set of values of the parameters N, n, γ, kd, β, and Θ. However, (3.43) is too complicated for a general discussion.

Some numerical results have been derived in the particular case N = 2, n = 1.5 (Fig.5.1)

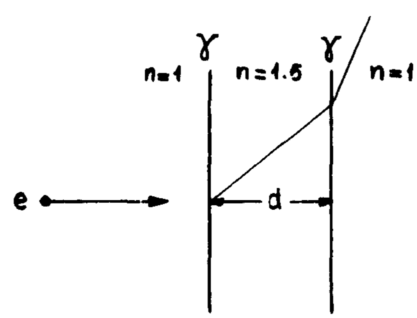


Fig.5.1 - The two-film target

5.1 - The gain in the Čerenkov direction over the system with $\gamma = 0$

The quantity $|R_o|^2/\sin^2\Theta$ was evaluated for a number of values of the conductivity of the films γ, of the spacing kd between films (measured in wavelengths) and of the velocity β of the

particle. For each value of β, Θ was made to assume the value corresponding to the direction of Čerenkov radiation. This is simply obtained by putting (*) $\beta n \cos\theta = \pm 1$ in (3.43). The corresponding values of Θ are derived from the equation (see for example (4.1-7))

$$\sin \Theta = n \sin\theta = n \sqrt{1 - \frac{1}{\beta^2 n^2}} \tag{5.1-1}$$

For $\beta n = \pm 1$ (that is for $|\beta| = 1/1.5$) one has $\Theta = 0$; for $\beta = \pm 1/\sqrt{n^2-1}$ (that is for $|\beta| = 1/\sqrt{1.25}$) one has $\Theta = \pi/2$.

The evaluation of (3.43) requires a limit to be performed, since $\check{F}$ has a pole at $\beta n \cos \theta = \pm 1$. The result is

$$\frac{|R_o|^2}{\sin^2 \Theta} = \left[\frac{eckZ_o}{4\pi^2 n(n^2-1)}\right]^2 \frac{\beta^2 n^2 - 1}{\beta^4} \frac{f_1(n,\eta,\phi,\beta,\delta)}{f_2(n,\eta,\phi,\delta)} \tag{5.1-2}$$

where

$$\delta = \pm \frac{\gamma Z_o}{2n^2\beta}$$

$$\eta = \pm \frac{1}{n} \frac{1}{\sqrt{1-\beta^2(n^2-1)}}$$

(the upper signs hold for back scattering, the lower signs for forward scattering) and

(*) Recall that back scattering corresponds formally to positive values of both $\cos \theta$ and β, while forward scattering corresponds to positive values of $\cos \theta$ and negative values of β.

$$(5.1\text{-}3)\quad f_1(n,\eta,\phi,\beta,\delta) = 64\,\delta^2 n^6\cos^2\phi + \beta^4(n^2-1)^2\,\phi^2(n-\eta-2\delta n)^2 +$$

$$+\sin^2\phi\left[(2-\beta^2)(n^2-1)(\eta+n+2\delta n) - 4\delta n^2(n-\eta-2\delta n)\right]^2 +$$

$$+\,2\beta^2(n^2-1)\left[n^2-(\eta+2\delta n)^2\right]\left[(2-\beta^2)(n^2-1) + 4\delta n^2\right]\phi\sin\phi\,\cos\phi$$

$$(5.1\text{-}4)\quad f_2(n,\eta,\phi,\delta) = 4n^2(\eta+2\delta n)^2\cos^2\phi + \left[(\eta+2n\delta)^2 + n^2\right]^2\sin^2\phi$$

From (5.1-2) (5.1-3) (5.1-4) one obtains immediately the gain G of the system shown in Fig.5.1 over an analogous system with $\gamma = 0$:

$$(5.1\text{-}5)\qquad G = \frac{f_1(n,\eta,\phi,\beta,\delta)\; f_2(n,\eta,\phi,0)}{f_2(n,\eta,\phi,\delta)\; f_1(n,\eta,\phi,\beta,0)}$$

where

$$(5.1\text{-}6)\quad f_1(n,\eta,\phi,\beta,0) = \beta^4(n^2-1)^2\phi^2(n-\eta)^2 + \sin^2\phi\,(2-\beta^2)^2(n^2-1)^2(\eta+n)^2 +$$

$$+\,2\beta^2(n^2-1)^2(2-\beta^2)(n^2-\eta^2)\,\phi\sin\phi\,\cos\phi$$

and

$$(5.1\text{-}7)\quad f_2(n,\eta,\phi,0) = 4n^2\eta^2\cos^2\phi + (n^2+\eta^2)^2\sin^2\phi$$

It is immediately seen from (5.1-5) (5.1-6) (5.1-7) that G tends to infinity for $\phi \to 0$, that is for kd $\to 0$. This was to be expected, since the radiation of a system with $\gamma = 0$ tends to vanish for $d \to 0$.

As regards the back-scattered radiation, the gain G is

generally larger than unity, due to the small reflecting properties of the system with $\gamma = 0$. It is interesting to note that, for given values of β and γ, G is a quasi-periodic function of kd, with high peaks corresponding to kd/β equal to a multiple of π. The conditions $\beta n \cos \Theta = 1$, $kd/\beta = m\pi$, (with m integral) yield $knd \cos \Theta_o = m\pi$, which is a resonance condition for the back-scattered radiation.

As to the forward radiation, we found values of G ranging from infinity to a limiting value smaller than unity (Fig.5.2 to Fig.5.5). The high values of G for small values of kd have already been explained. The value $G = 1$ occurs at about $kd/\beta = 2\pi$, quite independently of γ. Of course, the smaller γ, the wider are the ranges of the other variables corresponding to $G \simeq 1$.

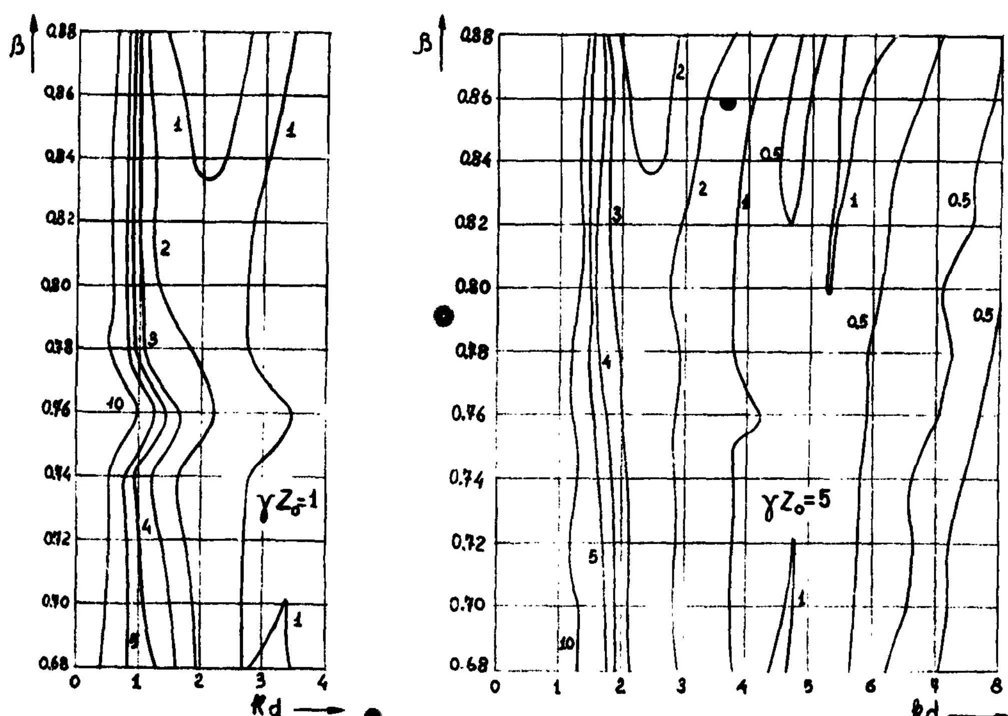


Fig.5.2 and 5.3 – Case N = 2, [illegible] = 1.[illegible] – Lines of constant gain in the Čerenkov direction, for $\gamma Z_o = 1$ and $\gamma Z_o = 5$ respectively, over a similar target with γZ_o = [illegible]

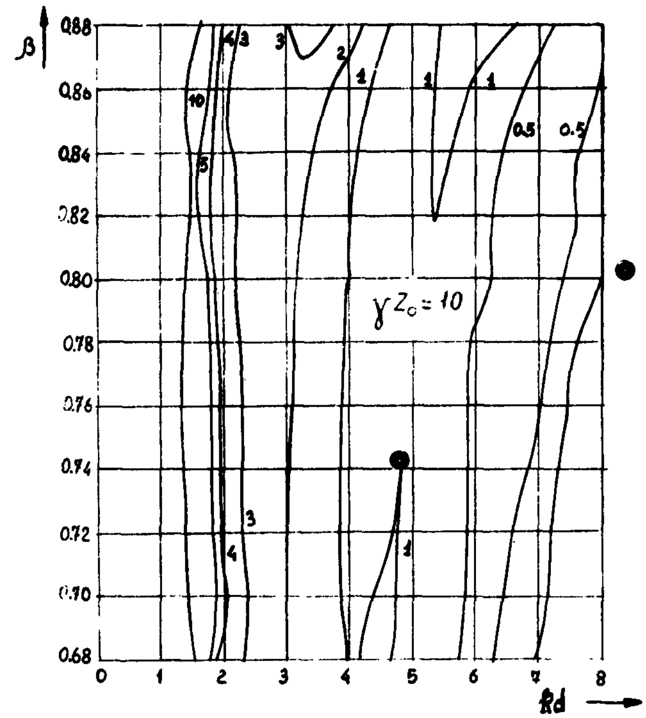


Fig.5.4

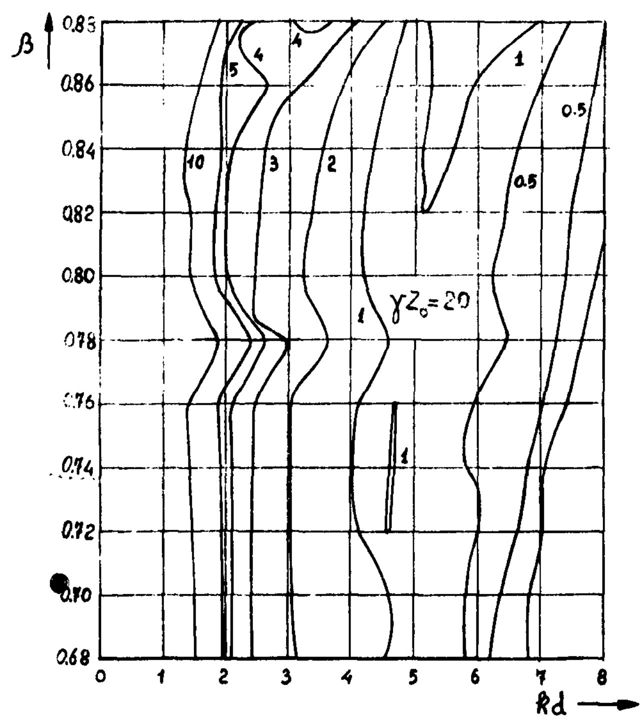


Fig.5.5

Fig.5.4 and 5.5 - Case N = 2, n = 1.5 - **Lines of constant gain in the Čerenkov direction, for γZ_o = 10 and γZ_o = 20 respectively, over a similar target with γZ_o = 0.**

5.2 - Forward radiation pattern from a two-film target.

The radiation pattern, that is $|R_0|^2/\sin^2\Theta$ versus Θ, has been evaluated in the particular cases $\beta = 0.8$; $\phi = \pi, 2\pi, 3\pi$. The quantity γZ_0 was made to assume the values 0, 1, 5, 10, 20 as in the preceding section. The results are shown in Figs.5.6 to 5.8. The Čerenkov direction corresponds to $\Theta \simeq 56^\circ$.

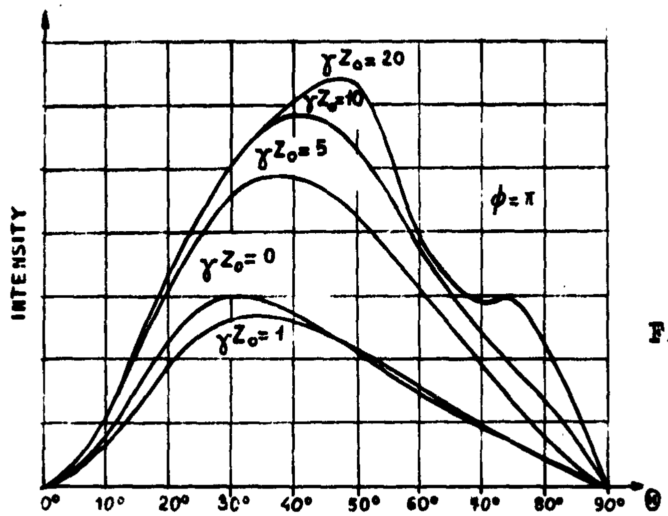


Fig.5.6

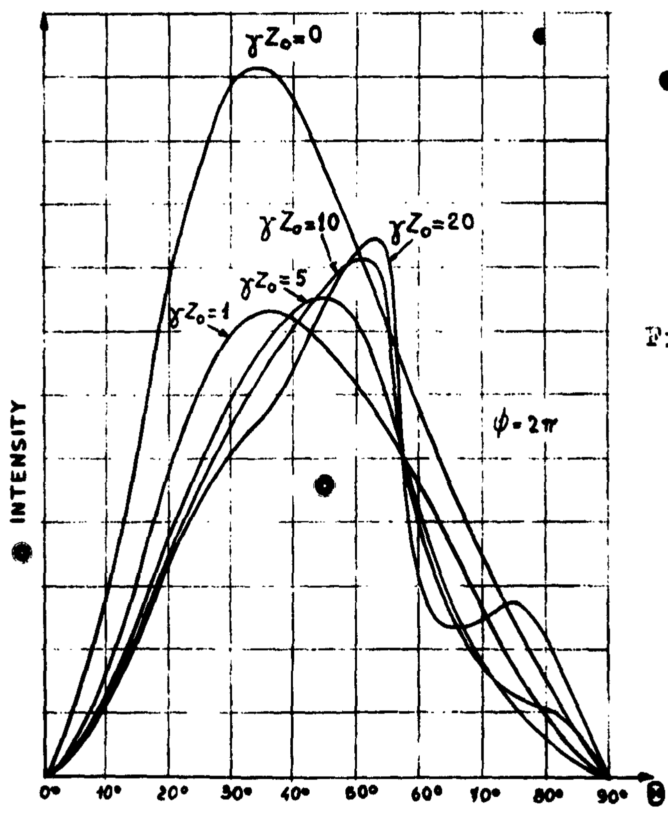


Fig.5.7

Fig.5.6 and 5.7 - Case N = 2, n = 1.5, $\beta = 0.8$ - Radiation pattern for some values of γZ_0, for $\phi = \pi$ and 2π respectively.

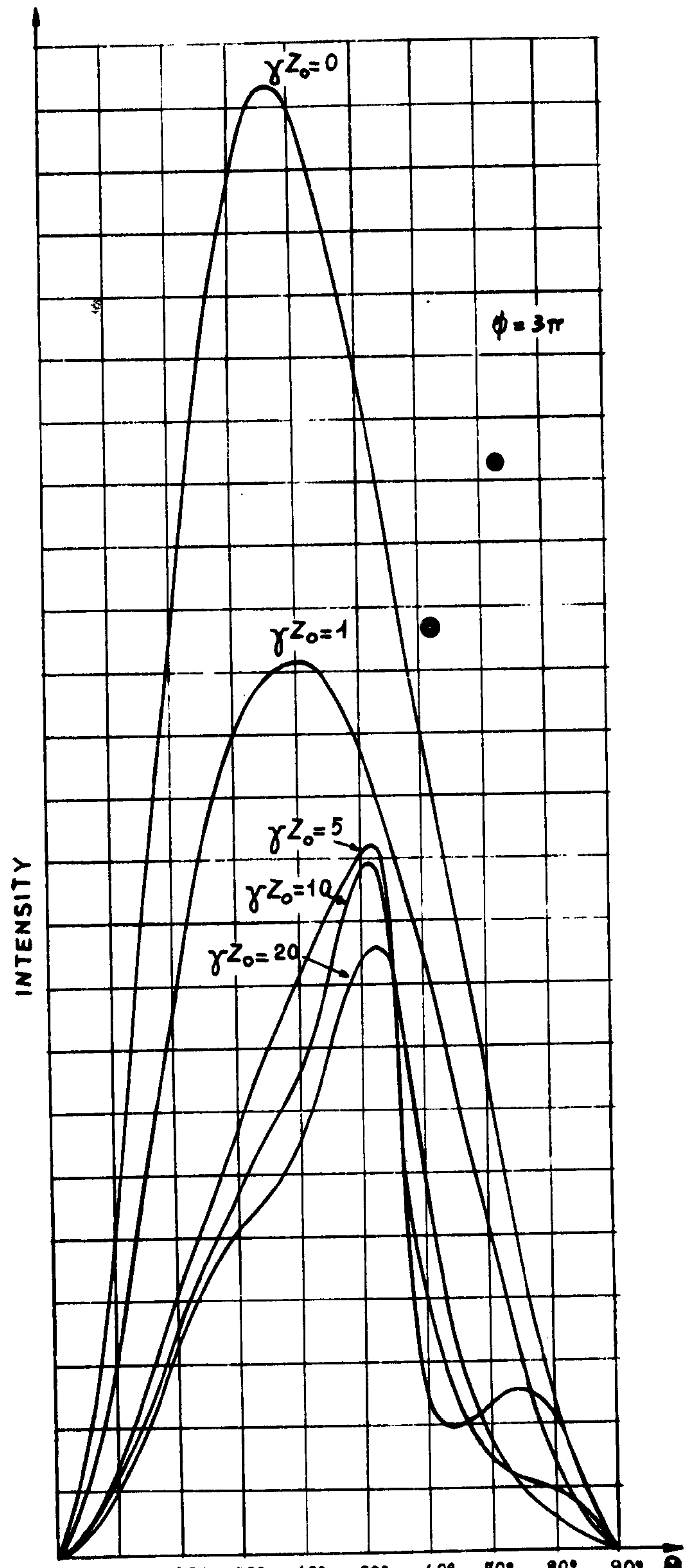


Fig.5.8 – Case N = 2, n = 1.5, β = 0.8 – Radiation pattern for some values of γZ_0, for $\phi = 3\pi$.

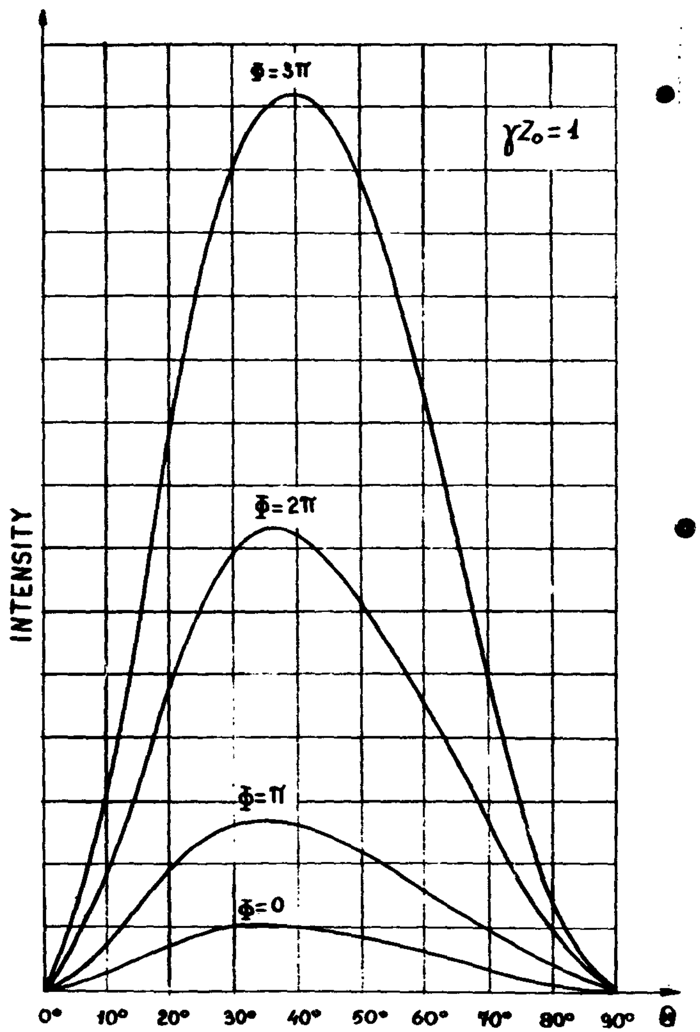
From Fig.5.6 it appears that at $\phi = \pi$ the intensity in any given direction is an increasing function of γ (except in the vicinity of $\gamma = 0$). This seems to indicate that at such a small spacing, Čerenkov radiation has but a small importance, compared with transition radiation which increases with γ.

Fig.5.7 shows that at $\phi = 2\pi$ the intensity is only slightly dependent on γ. This result generalizes the result precedingly found, that the gain G, defined by (5.1-5), is about 1 for $\phi \simeq 2\pi$, independently of γ, not only in the Čerenkov direction, but in all directions.

Fig.5.8 shows the behavior at $\phi = 3\pi$. Now the intensity is a more complicated function of γ. The over-all tendency is to be a decreasing function of γ. Generally speaking, one can say that Čerenkov radiation, though largely affected by diffraction, is now more important than transition radiation. However, by increasing γ, a larger part of power is lost by Joule effect on the films.

In order to investigate the dependence of the radiation pattern on the film spacing we can refer to Figs.5.9 to 5.12. It appears that at low conductivity ($\gamma Z_0 = 1$) the half-power beam-width is practically independent of ϕ. This result however is proved only for the small values of ϕ considered (recall that $\phi = 3\pi$ corresponds to $d < 3\lambda/2$, precisely, to $d = 1.2\lambda$).

By increasing γ, the radiation pattern becomes sharper in the vicinity of the Čerenkov direction, when ϕ increases. This effect is probably due to interference of the Čerenkov radiation with transition radiation.



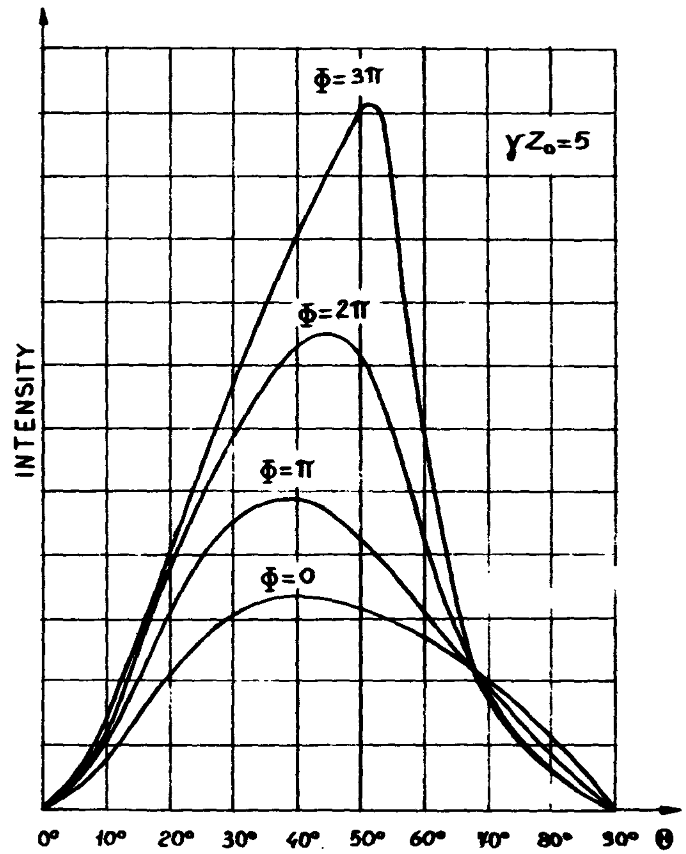


Fig.5.9 and 5.10 - **Case** $N = 2$, $n = 1.5$, $\beta = 0.8$ - The radiation **pattern** for some **values** of ϕ , for γZ_o equal to 1 and 5 respectively.

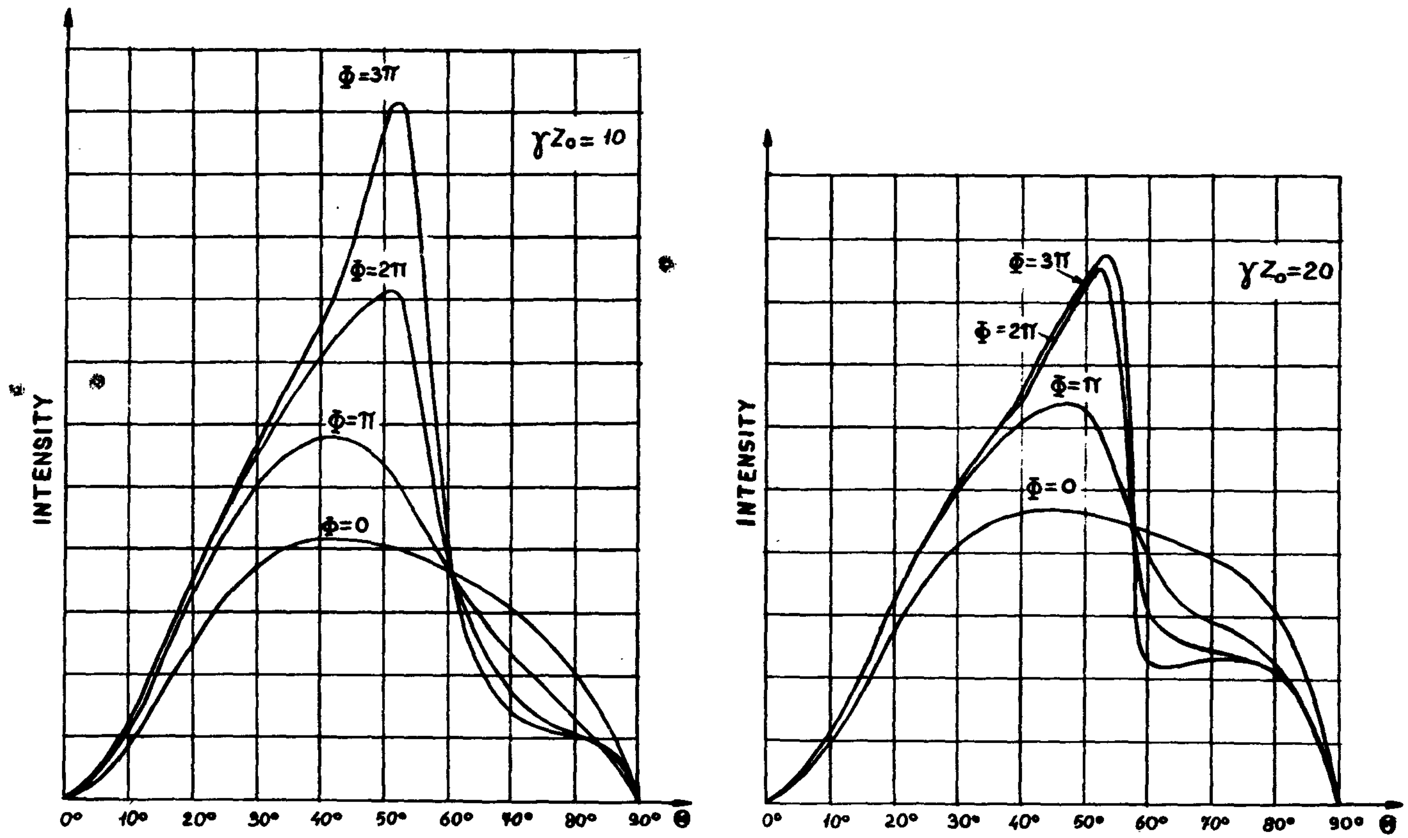


Fig.5.11 and 5.12 - Case N = 2, n = 1.5, β = 0.8 - The radiation pattern for some values of ϕ , for γZ_0 equal to 10 and 20 respectively.

5.3 - Forward radiation pattern from a ten-film target.

The radiation pattern has been evaluated in the case $N = 10$, $n = 1.5$, $\beta = 0.8$, for $\phi = \pi, 2\pi, 3\pi$ and $\gamma Z_0 = 0, 1, 5, 10, 20$. The angle Θ' was made to vary from 0° to 90° by 10° steps, except in the range 50° to 60°, where it varied by 1° steps. (Recall that the Čerenkov direction corresponds to $\Theta' \simeq 56^\circ$). In spite of this fine division of the Θ'-range, the radiation pattern cannot be plotted with great accuracy for $\gamma Z_0 \neq 0$, since it turns out that a still finer spacing would be needed. However, the following results can be given, which in any case must be understood as pessimistic.

1) for one and the same value of ϕ, the main lobe becomes weaker when the film conductivity increases (Fig. 5.13) . This is presumably due to the fact that the power lost by Joule effect on the films increases when the conductivity increases.

2) for one and the same value of ϕ, the main lobe becomes narrower when the film conductivity increases. In Fig. 5.14 we report as an example the main lobes corresponding to the various values of γZ_0, for $\phi = 3\pi$, with maxima normalized to unit. In the Figure, the values of the half-power beam-widths are indicated.

6 - Conclusion

Due to the complexity of the expression (3.43) of R_0, conclusions can be derived only in the two cases $N = 2$ and $N = 10$ for which numerical calculations have been carried out.

In the case $N = 2$, we have found that the effi-

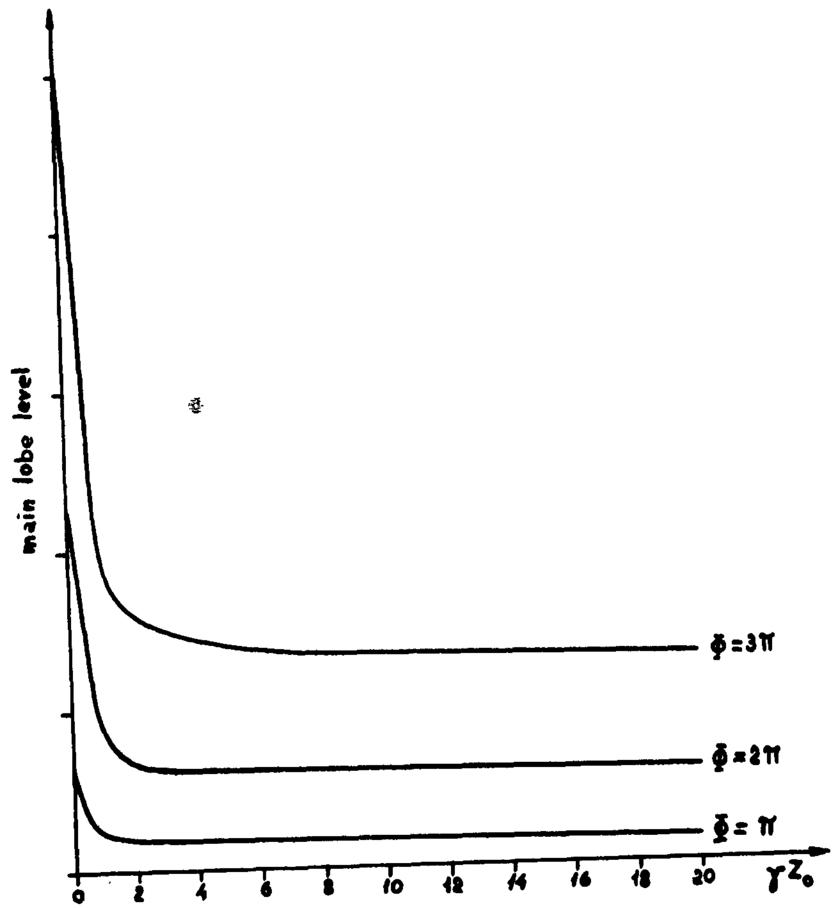


Fig.5.13 - Case N = 10, n = 1.5, β = 0.8 - The main lobe level versus γZ_o at $\phi = \pi$, 2π, 3π.

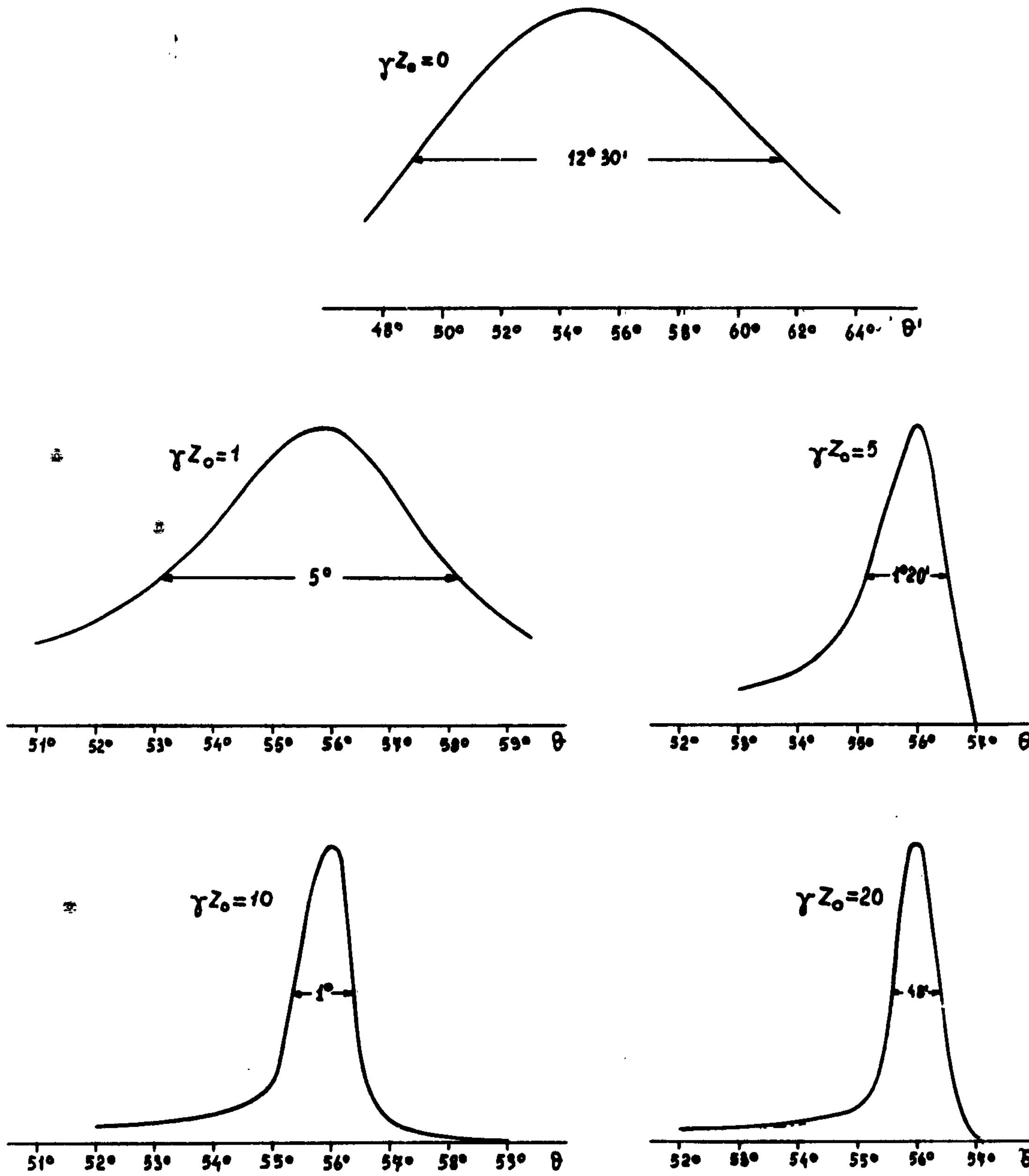


Fig. 5.14 – Case N = 10, n = 1.5, β = 0.8 – The main lobe for $\phi = 3\pi$, and some values of γZ_0.

ciency of a structure characterized by a given value of the film conductivity is better than the efficiency of an analogous structure with vanishing film conductivity,only if the thickness of the dielectric layer is less than one wavelength - a condition not easy to realize at submillimetric wavelengths.

A more encouraging result regards the half-power beam-width, which turned out to be a decreasing function of the film conductivity, at least in the range for which numerical calculations have been carried out .

In this connection, the results obtained when $N = 2$ are not indicative, since too small value of the overall thickness are involved and the effect of diffraction is therefore very pronounced. With $N = 10$, we found that, for $\phi = 3\pi$, the half-power beam-width varies from about 12 degrees to less than 1 degree when γZ_0 varies from 0 to 20.

Of course, in order to discuss the application of this result to practical problems, like the measurement of particle velocity, other numerical calculations would be required. For instance a determination of the radiation pattern for a number of values of β, and for fixed values of the other parameters , would be desiderable.

In conclusion, even the few numerical results obtained, seem to indicate that a system of the type investigated is not very suitable to produce submillimeter power with reasonable efficiency. In this connection the passage from the below-threshold case, discussed in a previous report ([1]), to the above-threshold case is not so favorable as might have

been surmised .

Any way, an interesting feature of the radiation in the above-threshold case is represented by the possibility to obtain comparatively narrow beam-widths at certain wavelengths. This property could perhaps be exploited for the measurement of particle velocities.